EFFECT OF RHEOLOGICAL FACTORS ON THE LAWS OF MOTION AND HEAT TRANSFER OF VISCOELASTIC FLOWS AT LOW DEBORAH NUMBERS

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The effect of structurally viscous and viscoelastic factors on the laws of motion and heat transfer in the entrance region of a channel are analyzed for the Deborah numbers.

The laws of motion and heat transfer of Newtonian fluids in the entrance region of channels have been thoroughly studied. Results in good agreement with experiment have been obtained by approximate and exact methods of solution [1].

The motion of viscoelastic fluids in the entrance region of channels has been treated in a number of papers. The main problem is the explanation of the effect of the rheological factors of fluids (the nonlinearity of the flow curve and the value of the reversible elastic deformation) on the pressure losses, the velocity distribution, the entrance lengths, etc.

Experiments with viscoelastic solutions of various concentrations for Reynolds numbers from 6 to 2000 led Sylvester and Rosen [2] to the conclusion that the nonlinearity of the flow curve (the exponent n in a power law) and the value of the reversible elastic deformation γ_e have opposite effects on the pressure losses in the entrance region of a pipe.

Brocklebank and Smith [3] used flow visualization to measure the velocity field in various cross sections of the entrance region and determined entrance lengths. They showed that entrance lengths for the flow of viscoelastic fluids are appreciably greater than for a Newtonian fluid. The entrance lengths increased with increasing elasticity of the solutions. Similar results follow from theoretical studies [4, 5].

Unfortunately, unanimity in these questions has not yet been achieved. Experiments [6] with viscoelastic solutions of various concentrations over a range of Reynolds numbers from 1 to 270 showed that entrance lengths were 10-100% shorter than those obtained with inelastic fluids for the same power-law parameters.

Tandon [7] for flat channels and Bilgen [8] for circular pipes, using approximate methods of boundary-layer theory, also concluded that the entrance length decreases with increasing elasticity.

In the present paper we use boundary-layer theory methods to investigate the effect of rheological factors (structurally viscous β_0 and viscoelastic γ_e) on the laws of motion and heat transfer of a fluid in the entrance region of a channel for the following velocity field.

$$u = u(x, y),\ v = v(x, y),\ w = 0. \quad (1)$$

In developing flows of viscoelastic fields, including those in the entrance region of channels, the first difference of the normal stresses $t_{xx} - t_{yy}$ is different from zero. Therefore, with the usual approximations of boundary-layer theory [1], the equations of motion of a viscoelastic fluid can be written in the form

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial t_{xx}}{\partial x} + \frac{\partial \tau}{\partial y},$$
$$0 = -\frac{\partial p}{\partial y} + \frac{\partial t_{yy}}{\partial y}. \quad (2)$$

Since τ and $t_{xx} - t_{yy}$ are zero in the flow core, Eqs. (1) reduce to the form

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$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-U\frac{\partial U}{\partial x}\right)=\frac{\partial}{\partial x}(t_{xx}-t_{yy})+\frac{\partial\tau}{\partial y}. \quad (3)$$

Here $\rho U(\partial U/\partial x) = (\partial t_{yy}/\partial x)_{core}$. Equation (3) is the starting point for the derivation of the integral momentum equation. Omitting terms known for ordinary Newtonian fluids, we obtain finally

$$\rho\left[U\frac{\partial U}{\partial x}\left(2\delta^{**}+\delta^{*}\right)+U^2\frac{\partial\delta^{**}}{\partial x}\right]=\tau_w(x)-\int_0^\delta\frac{\partial}{\partial x}(t_{xx}-t_{yy})\,dy. \quad (4)$$

Here

$$\delta^{*}=\frac{1}{U}\int_0^\delta(U-u)\,dy,\quad \delta^{**}=\frac{1}{U^2}\int_0^\delta u\,(U-u)\,dy.$$

In the stabilized flow region the velocity profile of a viscoelastic fluid obeying a linear fluidity law has the form [9]

$$\omega^0\equiv\frac{u}{U_{\max}}=A\left[2\xi-\xi^2+B\left(3\xi-\xi^2+\xi^3\right)\right],\ A=(1+B)^{-1}, \quad (5)$$

$$B=\frac{4}{9}\left[\left(1+18\frac{\beta_0}{Re_0}\right)^{0,5}-1\right],\quad \xi=\frac{y}{b},\quad \beta_0=\frac{\theta}{\varphi_0}\rho V^2.$$

Therefore, we approximate the velocity profile in the entrance region by a cubic polynomial

$$\omega\equiv u/U(x)=a_0+a_1k+a_2k^2+a_3k^3,\ k=y/\delta. \quad (6)$$

We find the coefficients from the boundary conditions $\omega = 0$ for $k = 0$; $\omega = 1$, $d\omega/dk = 0$ for $k = 1$:

$$\int_0^1\omega dk\Big|_{\delta(x)\to b}=\int_0^1\omega^0 d\xi.$$

Starting from these conditions, it can be shown that $a_0 = 0$, $a_1 = -6 + 8A + 9AB$, $a_2 = 15 - 16A - 18AB$, and $a_3 = -8 + 8A + 9AB$. Therefore,

$$\delta^{*}=\delta\int_0^1\left(1-\frac{u}{U(x)}\right)dk=A_1\delta, \quad (7)$$

$$\delta^{**}=\delta\int_0^1\frac{u}{U(x)}\left(1-\frac{u}{U(x)}\right)dk=B_1\delta, \quad (8)$$

$$A_1=1-\frac{a_1}{2}-\frac{a_2}{3}-\frac{a_3}{4},$$

$$B_1=\frac{a_1}{2}+\frac{a_2}{3}+\frac{a_3}{4}-\frac{a_1^2}{3}-\frac{a_2^2}{5}-\frac{a_3^2}{7}-\frac{a_1a_2}{2}-\frac{a_2a_3}{3}-\frac{2a_1a_3}{5}.$$

From the condition for a constant flow rate in a flat channel, assuming that the velocity profile is uniform at entry,

$$\delta\int_0^\delta u\,dy+U(x)(b-\delta)=Vb$$

we have

$$U(x)/V=(1-A_1\Delta)^{-1},\ \Delta=\delta(x)/b. \quad (9)$$

Using the fundamental rule for differentiating under the integral sign, and taking account of the fact that in the region where the boundary layer and the flow core join $t_{xx} - t_{yy} \approx 0$, Eq. (4) can be written in the form

$$\rho\left[U\frac{\partial U}{\partial x}(2\delta^{**}+\delta^{*})+U^2\frac{\partial\delta^{**}}{\partial x}\right]=\tau_w(x)-\frac{\partial}{\partial x}\int_0^\delta(t_{xx}-t_{yy})\,dy. \quad (10)$$

Substituting the appropriate expressions from (7)-(9) into (10), we obtain finally

$$\frac{B_1 + A_1(A_1+B_1)\Delta}{(1-A_1\Delta)^3} \cdot \frac{d\delta}{dx} = \frac{1}{\rho V^2}\left[\tau_{\text{ст}}(x) - \frac{d}{dx}\int_0^\delta (t_{xx}-t_{yy})\,dy\right]. \tag{11}$$

For Deborah numbers De ≪ 1 the relation between the symmetric stress and rate of strain tensors can be written in the form [10]

$$\tau_{ij} = -p\delta_{ij} + \mu_1(I_2)\dot{e}_{ij} + \mu_2(I_2)\dot{e}_{ij}^2 + \mu_3(I_2)\ddot{e}_{ij}. \tag{12}$$

Here $\dot{e}_{ij} \equiv \partial v_i/\partial x_j + \partial v_j/\partial x_i$ is the rate of strain tensor, $\ddot{e}_{ij} \equiv \delta\dot{e}_{ij}/\delta t = v_m \partial\dot{e}_{ij}/\partial x_m \cdot \partial v_i/\partial x_m \cdot \dot{e}_{mj}\partial v_i/\partial x_m - \dot{e}_{jm}$ is the acceleration tensor for steady flow, and the $\mu_i(I_2)$ are experimentally determined properties of the viscoelastic field [scalar functions of the second invariant of the rate of strain tensor, in our case $I_2 = (du/dy)^2$, or of the second invariant of the stress tensor $T_2 = \tau^2$].

Using (12) and making the usual approximations of boundary-layer theory, the components of the stress tensor appearing in Eq. (11) for the velocity field (1) have the form

$$\tau_w(x) = \mu_1(I_2)\left(\frac{\partial u}{\partial y}\right)_w, \quad t_{xx} - t_{yy} = \mu_3(I_2)\left(\frac{\partial u}{\partial y}\right)^2.$$

Assuming [9] that the fluidity $[\mu_1(I_2)]^{-1} \equiv \varphi(T_2) = \varphi_0 + \theta T_2^{1/2}$, and $\mu_3/\mu_1 = \lambda$ is the characteristic relaxation time of the fluid, we can write

$$\tau(x) = \frac{\varphi_0}{2\theta}\left[\left(1 + \frac{4\theta}{\varphi_0^2}\cdot\frac{\partial u}{\partial y}\right)^{0,5} - 1\right],$$

$$t_{xx} - t_{yy} = \gamma_e \tau(x), \quad \gamma_e = \lambda\left\langle\frac{\partial u}{\partial y}\right\rangle \approx \frac{3\lambda V}{2b}.$$

In accord with (6)

$$\tau(x) = \frac{\varphi_0}{2\theta}\left[\left(1 + \frac{8\beta_0}{\text{Re}_0}\cdot\frac{a_1 + 2a_2k + 3a_3k^2}{\Delta(1-A_1\Delta)}\right)^{0,5} - 1\right]. \tag{13}$$

For small values of $\beta_0/\text{Re}_0 \le 0.04$ we can limit ourselves to three terms in the expansion in (13). From (11) an expression can be obtained for the dimensionless entrance length

$$\frac{1}{\text{Re}_0}\cdot\frac{x}{d} = \frac{A_1+B_1}{4a_1A_1}\Bigg\{\Delta + \Bigg[\frac{2A_1+3B_1}{2}(F(\Delta)-F(0)) - \frac{1}{\sqrt{7}}\Bigg(\frac{A_1+2B_1}{a_1A_1\frac{\beta_0}{\text{Re}_0}} - 2A_1 - 3B_1\Bigg)(\Phi(\Delta)-\Phi(0))\Bigg]\frac{1}{2A_1(A_1+B_1)} + \frac{1}{A_1R}\Bigg[\Bigg(\frac{4A_1+7B_1}{4A_1(A_1+B_1)} - a_1\frac{\beta_0}{\text{Re}_0}\Bigg)\Bigg(\frac{\cos\Pi}{2}(S(\Delta)-S(0)) - \sin\Pi(W(\Delta)-W(0)) + \frac{1}{\sqrt{7}}\Bigg(\frac{A_1+2B_1}{4a_1\frac{\beta_0}{\text{Re}_0}A_1^2(A_1+B_1)} - 3a_1\frac{\beta_0}{\text{Re}_0} - \frac{4A_1+7B_1}{4A_1(A_1+B_1)}\Bigg)\times\Bigg(\frac{\sin\Pi}{2}(S(\Delta)-S(0)) + \cos\Pi(W(\Delta)-W(0))\Bigg)\Bigg]\Bigg\}, \tag{14}$$

$$\Pi = -\frac{1}{2}\,\text{arctg}\,\frac{\sqrt{7}}{\dfrac{1}{4a_1A_1\frac{\beta_0}{\text{Re}_0}} - 1},$$

$$R = \sqrt[4]{\frac{1}{16A_1^4} - \frac{a_1}{2A_1^3}\cdot\frac{\beta_0}{\text{Re}_0} + 8\left(\frac{a_1}{A_1}\cdot\frac{\beta_0}{\text{Re}_0}\right)^2},$$

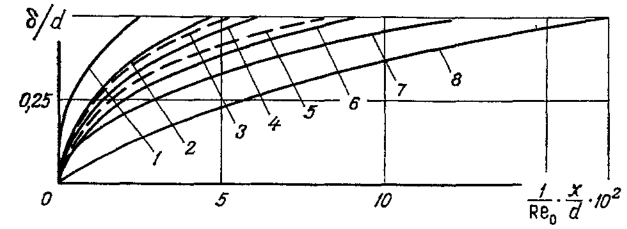

Fig. 1. Variation of boundary layer thickness with length: 1) $\beta_0/Re_0 = 0$, $\gamma_e/Re_0 = 0$; 2) 1.0, 0; 3) 0, 0.5; 4) 2.0, 0; 5) 0, 1.0; 6) 5.0, 0; 7) 10.0, 0; 8) 5.0, 0.5.

$$F(\Delta)=\ln\left[\left(\Delta^2-\frac{\Delta}{A_1}+\frac{a_1}{A_1}\cdot\frac{\beta_0}{Re_0}\right)^2+7\left(\frac{a_1}{A_1}\cdot\frac{\beta_0}{Re_0}\right)^2\right],$$

$$\Phi(\Delta)=\text{arctg}\left(\frac{\sqrt{7}\,a_1\frac{\beta_0}{Re_0}}{A_1\left(\Delta^2-\frac{\Delta}{A_1}+\frac{a_1}{A_1}\cdot\frac{\beta_0}{Re_0}\right)}\right),$$

$$S(\Delta)=\ln\frac{\left(\Delta-\frac{1}{2A_1}\right)^2-2R\cos\Pi\left(\Delta-\frac{1}{2A_1}\right)+R^2}{\left(\Delta-\frac{1}{2A_1}\right)^2+2R\cos\Pi\left(\Delta-\frac{1}{2A_1}\right)+R^2},$$

$$W(\Delta)=\text{arctg}\frac{2R\sin\Pi\left(\Delta-\frac{1}{2A_1}\right)}{\left(\Delta-\frac{1}{2A_1}\right)^2-R^2}.$$

Calculations with Eq. (14) showed that the entrance length increases with increasing β_0/Re_0, the characteristic of the nonlinearity of the flow curve.

For values of $\beta_0/Re_0 \geq 1$ the expression for the tangential stress can be written in the form

$$\tau(x)\approx\frac{\varphi_0}{2\theta}\sqrt{8\frac{\beta_0}{Re_0}\cdot\frac{a_1+2a_2k+3a_3k^2}{\Delta(1-A_1\Delta)}}.$$

As a consequence of this the expression for the entrance length takes the simpler form

$$\frac{1}{Re_0}\cdot\frac{x}{d}=\sqrt{\frac{\beta_0}{Re_0}}\frac{1}{\sqrt{2a_1}}\left\{\frac{B_1}{3}\left(\frac{\Delta}{1-A_1\Delta}\right)^{3/2}+(A_1+B_1)\left[\frac{1}{3}\left(\frac{\Delta}{1-A_1\Delta}\right)^{3/2}-\frac{1}{A_1}\left(\frac{\Delta}{1-A_1\Delta}\right)^{1/2}-\frac{1}{A_1^{3/2}}\left(\text{arctg}\sqrt{\frac{1-A_1\Delta}{A_1\Delta}}-\frac{\pi}{2}\right)\right]\right\}-\frac{\gamma_e}{Re_0}\cdot\frac{\ln(1-A_1\Delta)}{4A_1}\left[\frac{3a_3a_1-a_2^2}{6a_3\sqrt{3a_3a_1}}\ln\frac{3a_3+a_2}{\sqrt{3a_3a_1}+a_2}-\frac{a_2}{6a_3}\right]. \qquad (15)$$

Using Eq. (15) the development of the boundary layer (Fig. 1) and the local frictional drag coefficients (Fig. 2) were calculated for various of β_0/Re_0 and γ_e/Re_0. It is clear that the relative frictional drag coefficient c_f/c_{f0} increases with increasing magnitude of the reversible elastic deformation γ_e, but decreases with increasing β_0.

Rheological fluids as a rule are characterized by Prandtl numbers $Pr \gg 1$. Thus, it can be assumed that the whole heat-transfer process in such fluids is confined to a narrow region near the wall where the velocity gradient is constant.

The system of equations for the thermal boundary layer is

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{\partial^2 T}{\partial y^2},\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

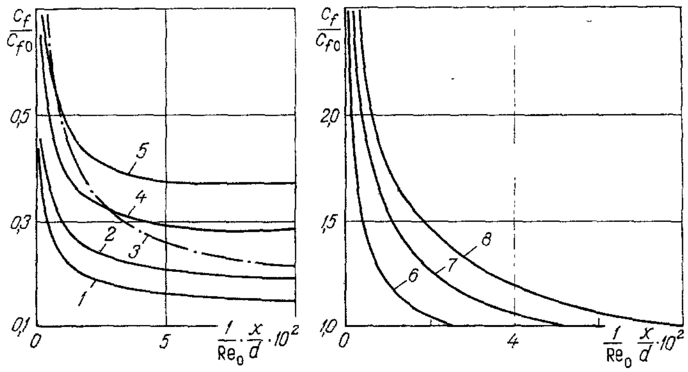

Fig. 2. Variations of relative frictional drag coefficients: 1) $\beta_0/Re_0 = 10.0$, $\gamma_e/Re_0 = 0$; 2) 5.0, 0; 3) 5.0, 0.5; 4) 2.0, 0; 5) 1.0, 0; 6) $\beta_0/Re_0 = 0$, $\gamma_e/Re_0 = 0$; 7) 0, 0.5; 8) 0, 1.0.

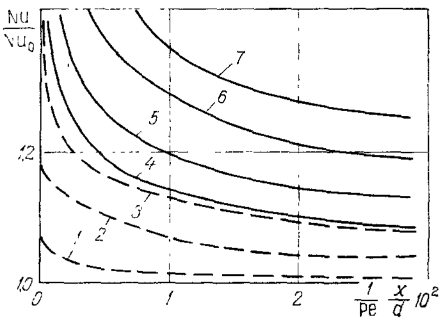

Fig. 3. Variation of relative heat-transfer coefficient: 1) $\beta_0/Re_0 = 0$, $\gamma_e/Re_0 = 0.1$; 2) 0, 0.5; 3) 0, 1.0; 4) 1.0, 0; 5) 2.0, 0; 6) 5.0, 0; 7) 10.0, 0.

The solution of these equations together with the linearized relation (6) for constant wall temperature can be written in the form [11]

$$Nu_x = 2b\sqrt{G}\left(\int_0^x \sqrt{G}\, d\xi\right)^{-1/3} \Big/ \left(\int_0^\infty \exp(-k^3/9)\, dk\right),$$

$$G = \frac{1}{a}\cdot\frac{\partial u}{\partial y}\bigg|_{y=0}.$$

Figure 3 shows the results of the calculation of the relative heat-transfer coefficient. The absolute values of β_0/Re_0 and γ_e/Re_0 have a pronounced effect on Nu/Nu_0. As one would expect for a fluid with $Pr \gg 1$, the rheological factors have a relatively small effect on the heat-transfer coefficient.

NOTATION

x, y	are the longitudinal and transverse coordinates;
u, v	are the longitudinal and transverse velocity components;
$b = d/2$	is the halfheight of channel;
$\delta(x)$	is the boundary-layer thickness;
U(x)	is the centerline velocity;
V	is the average velocity over cross section of channel;
φ_0	is the fluidity as $\tau \to 0$;
θ	is the parameter characterizing structural properties of fluid in linear fluidity law;
De	is the Deborah number;
Pr	is the Prandtl number;
Nu_x	is the local Nusselt number;
Nu_0	is the local Nusselt number for a Newtonian fluid;
$Re_0 = 2\rho\varphi_0 bV$	is the Reynolds number;

$c_f = 2\tau_w/\rho V^2$	is the frictional drag coefficient;
c_{f0}	is the frictional drag coefficient of Newtonian fluid in stabilized flow region.

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THERMOCONVECTION WAVES IN ASYMMETRICAL FLUIDS

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The propagation of thermoconvection waves in asymmetrical fluids is investigated. The results lead to a number of conclusions about the influence of microinertia and couple stresses on the wave propagation velocity and damping.

Lykov and Berkovskii [1, 2] have investigated the propagation of thermoconvection waves in viscous and viscoelastic fluids. Listrov and Shurinov [3] have studied the propagation of small shear disturbances in certain asymmetrical media. Here we consider the propagation of thermoconvection waves in asymmetrical fluids, using the equations of motion with regard for compressibility in the form [4, 5]

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0, \tag{1}$$

$$\rho\frac{d\mathbf{v}}{dt} = -\operatorname{grad} p + k\operatorname{rot}\boldsymbol{\omega} - (\mu + k)\operatorname{rot}\operatorname{rot}\mathbf{v} + (\lambda + 2\mu + k)\operatorname{grad}\operatorname{div}\mathbf{v} + \rho\mathbf{g}, \tag{2}$$

$$\rho J\frac{d\boldsymbol{\omega}}{dt} = -2k\boldsymbol{\omega} + k\operatorname{rot}\mathbf{v} - \gamma\operatorname{rot}\operatorname{rot}\boldsymbol{\omega} + (\alpha + \beta + \gamma)\operatorname{grad}\operatorname{div}\boldsymbol{\omega}. \tag{3}$$

The tensile stresses t_{ij} and couple stresses m_{ij} are determined from the rheological equations

$$t_{ij} = (-p + \lambda\operatorname{div}\mathbf{v})\,\delta_{ij} + \left(\mu + \frac{k}{2}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + k\varepsilon_{ijm}\left(\frac{1}{2}\varepsilon_{mrl}\frac{\partial v_l}{\partial x_r} - \omega_m\right), \tag{4}$$

$$m_{ij} = \alpha(\operatorname{div}\boldsymbol{\omega})\,\delta_{ij} + \beta\frac{\partial\omega_i}{\partial x_j} + \gamma\frac{\partial\omega_j}{\partial x_i}. \tag{5}$$

We write the heat-transfer equation in the form [6]

$$\rho c_p\left(\frac{\partial T}{\partial t} + v_k\frac{\partial T}{\partial x_k}\right) = \theta\Delta T + R, \tag{6}$$

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